Exact closed-form solutions of the Dirac equation with a scalar exponential potential

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Abstract

The problem of a fermion subject to a general scalar potential in a two-dimensional world for nonzero eigenenergies is mapped into a Sturm-Liouville problem for the upper component of the Dirac spinor. In the specific circumstance of an exponential potential, we have an effective Morse potential which reveals itself as an essentially relativistic problem. Exact bound solutions are found in closed form for this problem. The behaviour of the upper and lower components of the Dirac spinor is discussed in detail, particularly the existence of zero modes.

The Coulomb potential of a point electric charge in a 1+1 dimension, considered as the time component of a Lorentz vector, is linear ($\sim x$) and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [1]-[2] and in the Thirring-Schwinger model [3]. It is frustrating that, due to the tunneling effect (Klein´s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [4]-[5]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [6]-[7]. Recently it was incorrectly concluded that even in this case there is solely one bound state [8]. Later, the proper solutions for this last problem were found [9]-[11]. The mixed vector-scalar potential has also been analyzed for a linear potential [12] as well as for a general potential which goes to infinity as $x \to \infty$ [13]. In both of those last references it has been concluded that there is confinement if the scalar coupling is of sufficient intensity compared to the vector coupling.

The problem of a particle subject to an inversely linear potential in one spatial dimension ($\sim |x|^{-1}$), known as the one-dimensional hydrogen atom, has also received considerable attention in the literature. This problem presents some conundrums and the most perplexing one is regarding to the ground state. The nonrelativistic Schrödinger equation provides a ground-state solution with infinite eigenenergy and a related eigenfunction given by a delta function centered about the origin. This problem was also analyzed with the Klein-Gordon equation and there it was revealed a finite eigenenergy and an exponentially decreasing eigenfunction [14]. The problem was also approached with the Schrödinger, Klein-Gordon and Dirac equations and it was concluded that the Klein-Gordon equation provides unacceptable solutions while the Dirac equation, with the interacting potential considered as a time component of a vector, has no bounded solutions at all [15]. This problem was also sketched for a Lorentz scalar interacting potential in the Dirac equation [16], but the analysis is incomplete. In a recent paper [17], it was shown that the problem of a fermion under the influence of a general scalar potential for nonzero eigenenergies can be mapped into a Sturm-Liouville problem. Next, the key conditions for the existence of bound-state solutions were settled for power-law potentials. In addition, the solution for an inversely linear potential, including the zero-eigenmode, was obtained in closed form.

The problem of a fermion subject to a well potential given either by an exponential function ($V = V_0 \exp(-ar)$) is exactly solved for S-wave bound-

states in the nonrelativistic quantum mechanics [18]. On the other side, the problem for S-states for the Morse potential ($V = V_0\left[1-\exp(-ar)\right]^2$) is transformed into the difficult problem of solving a transcendent equation which is only approximately solvable [18]-[19]. The first problem presents some interest to the nucleon-nucleon system due to the short-range character of the interaction [20]-[21], whereas the second one has been used to describe the vibrations of nuclei in homonuclear diatomic molecules [22]-[23]. The one-dimensional asymmetric Morse potential ($-\infty < r < \infty$) is also an exactly solvable problem in the nonrelativistic quantum mechanics [24]-[25], even if its parameters are complex numbers [26]. In the present paper we approach the time-independent Dirac equation in 1+1 dimensions with a scalar potential given by an exponential potential. For nonzero eigenenergies this sort of potential gives rise to an effective Morse potential in a Sturm-Liouville problem for the upper component of the Dirac spinor. We find the bound states and discuss the existence of zero modes, which are related to the ultrarelativistic limit of the Dirac equation and play an important role in the induction of a fractional fermion number on the vacuum in the second quantization setting.

The two-dimensional Dirac equation can be obtained from the four-dimensional one with the mixture of spherically symmetric scalar, vector and anomalous magnetic-like (tensor) interactions. If we limit the fermion to move in the x-direction ($p_y = p_z = 0$) the four-dimensional Dirac equation decomposes into two equivalent two-dimensional equations with 2-component spinors and 2×2 matrices [27]. Then, there results that the scalar and vector interactions preserve their Lorentz structures whereas the anomalous magnetic interaction turns out to be a pseudoscalar interaction. Furthermore, in the 1+1 world there is no angular momentum so that the spin is absent. Therefore, the 1+1 dimensional Dirac equation allow us to explore the physical consequences of the negative-energy states in a mathematically simpler and more physically transparent way.

In the presence of a time-independent scalar potential the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass m reads

$$\left[c\alpha p + \beta\left(mc^2 + V\right)\right]\psi = E\psi, \tag{1}$$

where E is the energy of the fermion, c is the velocity of light and p is the momentum operator. α and β are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. One can choose the 2×2 Pauli matrices

satisfying the same algebra as α and β, resulting in a 2-component spinor ψ. The positive definite function $|\psi|^2 = \psi^\dagger \psi$, satisfying a continuity equation, is interpreted as a position probability density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states [28]. Using $\alpha = \sigma_2$, $\beta = \sigma_1$ and provided that the spinor is written in terms of the upper and the lower components

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{2}$$

the Dirac equation decomposes into :

$$E\phi = \left(mc^2 + V\right)\chi - \hbar c \chi', \tag{3}$$

$$E\chi = \left(mc^2 + V\right)\phi + \hbar c \phi', \tag{4}$$

where the prime denotes differentiation with respect to x. In terms of ϕ and χ the spinor is normalized as $\int_{-\infty}^{+\infty} dx \left(|\phi|^2 + |\chi|^2\right) = 1$, so that ϕ and χ are square integrable functions. It is remarkable that the Dirac equation with a scalar potential is not invariant under $V \rightarrow V + const$. Therefore, the absolute values of the energy will have physical significance and the freedom to choose a zero-energy will be lost.

Furthermore, using the expression for χ obtained from (4), viz.

$$\chi = \frac{\left(mc^2 + V\right)\phi + \hbar c\, \phi'}{E}, \quad E \neq 0, \tag{5}$$

and inserting it in (3) one arrives at the following second-order differential equation for ϕ:

$$-\frac{\hbar^2}{2}\, \phi'' + V_{eff}\, \phi = E_{eff}\, \phi, \tag{6}$$

where

$$E_{eff} = \frac{E^2 - m^2c^4}{2c^2} \tag{7}$$

$$V_{eff} = \frac{V^2}{2c^2} + mV - \frac{\hbar}{2c} V'. \tag{8}$$

The energy levels are symmetrical about $E = 0$ (see, *e.g.*, Refs. [29] and [30]). This conclusion can be obtained directly from (7). Besides, if ψ is a solution with energy E then $\sigma_3\psi^*$ is also a solution with energy $-E$ for the very same potential. It means that the potential couples to the positive-energy component of the spinor in the same way it couples to the negative-energy component. In other words, this sort of potential couples to the mass of the fermion instead of its charge so that there is no atmosphere for the production of particle-antiparticle pairs. No matter the intensity and sign of the coupling parameter, the positive- and the negative-energy solutions never meet. Thus there is no room for transitions from positive- to negative-energy solutions. This all means that Klein´s paradox never comes to the scenario.

Up to this point we have considered solutions for $E \neq 0$. Nevertheless, one could also ask for possible zero-energy solutions. These zero-mode solutions can be obtained directly from the Dirac equation (3)-(4). One can observe that the Dirac Hamiltonian with a general scalar potential always supports a zero-mode solution with upper and lower components given by

$$\phi = N_\phi \exp\left[-\left(\frac{mc^2x + v(x)}{\hbar c}\right)\right] \tag{9}$$

$$\chi = N_\chi \exp\left[+\left(\frac{mc^2x + v(x)}{\hbar c}\right)\right], \tag{10}$$

where N_ϕ and N_χ are constants and

$$v(x) = \int^x V(y)\, dy. \tag{11}$$

One can check that it is impossible to have both components different from zero simultaneously as physically acceptable solutions. A normalizable zero-mode eigenstate with a nonzero upper component for a massive fermion is possible only if $v(x)$ has a leading asymptotic behaviour given by

$$\lim_{x\to+\infty} v(x) \sim \left\{ \begin{array}{c} \text{constant} \\ \text{or} \\ +\infty \\ \text{or} \\ \pm|x|^{1-\delta}\,, \quad 0<\delta<1 \\ \text{or} \\ -\Delta|x|,\ -\infty\ <\Delta<mc^2 \\ \text{or} \\ -k\ln|x|,\ (k>0) \end{array} \right. \tag{12}$$

and

$$\lim_{x\to-\infty} v(x) \sim \left\{ \begin{array}{c} |x|^{1+\varepsilon},\ \varepsilon>0 \\ \text{or} \\ +\Delta|x|,\ \Delta>mc^2. \end{array} \right. \tag{13}$$

On the other hand, a normalizable zero-mode eigenstate with a nonzero lower component for a massive fermion is possible only if one of the following restrictions below is satisfied

$$\lim_{x\to+\infty} v(x) \sim \left\{ \begin{array}{c} -|x|^{1+\varepsilon},\ \varepsilon>0 \\ \text{or} \\ -\Delta|x|, \quad \Delta>mc^2, \end{array} \right. \tag{14}$$

and

$$\lim_{x\to-\infty} v(x) \sim \left\{ \begin{array}{c} \text{constant} \\ \text{or} \\ -\infty \\ \text{or} \\ \pm|x|^{1-\delta}\,, \quad 0<\delta<1 \\ \text{or} \\ +\Delta|x|,\ -\infty\ <\Delta<mc^2 \\ \text{or} \\ +k\ln|x|,\ (k>0). \end{array} \right. \tag{15}$$

Notice that for a massless fermion it is enough that $v(x)$ grows faster than $\hbar c|x|$ ($-\hbar c|x|$) as $|x|\to\infty$ to have a nonzero upper (lower) component.

Incidentally, the existence of Dirac eigenspinors with a vanishing lower component, or even with a vanishing upper component, is due to the particular representations of the matrices α and β adopted in this paper. It

is instructive at this point to consider for a moment a representation where the eigenspinor presents a more familiar behaviour. Let us write the Dirac equation (1) as

$$\left[c\sigma_1 p+\sigma_3\left(mc^2+V\right)\right]\tilde{\psi}=E\tilde{\psi}. \tag{16}$$

The original spinor is related to $\tilde{\psi}$ by the unitary transformation $\psi=U\tilde{\psi}$, where

$$U=\frac{1}{\sqrt{2}}\left(\begin{array}{cc} 1 & -i \\ 1 & +i \end{array}\right), \tag{17}$$

so that $\phi=\left(\tilde{\phi}-i\tilde{\chi}\right)/\sqrt{2}$ and $\chi=\left(\tilde{\phi}+i\tilde{\chi}\right)/\sqrt{2}$. In the nonrelativistic approximation (16) becomes

$$\tilde{\chi}=\frac{p}{2mc}\,\tilde{\phi} \tag{18}$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right)\tilde{\phi}=\left(E-mc^2\right)\tilde{\phi}. \tag{19}$$

Eq. (18) shows that $\tilde{\chi}$ is of order $v/c<<1$ relative to $\tilde{\phi}$ and Eq. (19) shows that $\tilde{\phi}$ obeys the Schrödinger equation with the potential V. Now one can see that when one uses the representation where $\alpha=\sigma_2$, $\beta=\sigma_1$ (that one used in this paper) one obtains upper and lower components approximately equal to each other in the nonrelativistic limit. On the other side, in the ultrarelativistic limit one expects that $\tilde{\chi}$ presents a contribution comparable to $\tilde{\phi}$, thus the possibilities $\tilde{\phi}\approx -i\tilde{\chi}$ and $\tilde{\phi}\approx +i\tilde{\chi}$ imply into $\chi\approx 0$ and $\phi\approx 0$, respectively. Therefore, one can conclude that the zero-mode solutions correspond to the ultrarelativistic limit of the theory.

Now let us focus our attention on a nonconserving-parity scalar potential in the form

$$V(x)=A\exp\left(-\frac{\lambda}{\hbar c}x\right), \tag{20}$$

where λ is a positive parameter related to the range of the interaction.

It follows from Eqs. (12)-(15) that only when $A<0$, this potential meets the requirement to hold a zero-eigenmode. In passing, the lower component of the Dirac spinor vanishes. In terms of ξ and μ defined by

$$\xi \;=\; 2\frac{|A|}{\lambda}\exp\left(-\frac{\lambda}{\hbar c}x\right)$$

$$\mu = \frac{\sqrt{m^2c^4 - E^2}}{\lambda}, \tag{21}$$

the normalizable Dirac spinor corresponding to $E = 0$ is, then

$$\psi = N\,\xi^{\mu}\,e^{-\xi/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{22}$$

where N is a normalization constant. Note that the normalization of this spinor is possible only if $\mu > 0$, then the fermions must be massive. Once this criterion is verified the zero-eigenmode solution always exists for any negative value of A, regardless the mass of the fermion.

For $E \neq 0$, the potential (20) gives rise to the effective potential

$$V_{eff}(x) = V_1 \exp\left(-2\frac{\lambda}{\hbar c}x\right) + V_2 \exp\left(-\frac{\lambda}{\hbar c}x\right), \tag{23}$$

where

$$\begin{aligned} V_1 &= \frac{A^2}{2c^2} > 0 \\ V_2 &= \frac{\lambda A}{c^2}\left(\frac{mc^2}{\lambda} + \frac{1}{2}\right). \end{aligned} \tag{24}$$

The effective potential tends to $+\infty$ as $x \to -\infty$ and vanishes as $x \to +\infty$. Furthermore there is a local minimum if $V_2 < 0$. This condition for the existence of a local minimum requires that $A < 0$. The effective Morse potential with a well structure might hold bound states in spite of the fact that the original exponential potential given by (20) is everywhere repulsive, so that one can not expect bound solutions in the nonrelativistic limit. This essentially relativistic solution is due to the peculiar coupling in the Dirac equation: the scalar potential behaves like an x-dependent rest mass [28]. Using (6)-(8) one obtains the equation

$$\xi\phi'' + \phi' + \left(-\frac{\xi}{4} - \frac{\mu^2}{\xi} + \rho\right)\phi = 0, \tag{25}$$

where ρ is defined by

$$\rho = \frac{mc^2}{\lambda} + \frac{1}{2},$$

and the prime denotes differentiation with respect to ξ. The normalizable asymptotic form of the solution as $\xi \to \infty$ is $e^{-\xi/2}$. As $\xi \to 0$, when the term $1/\xi$ dominates, the regular solution behaves as ξ^{μ} and the solution for all ξ can be expressed as $\phi(\xi) = \xi^{\mu} e^{-\xi/2} w(\xi)$, where w is solution of the confluent hypergeometric equation [31]

$$\xi w'' + (b - \xi) w' - a w = 0, \tag{26}$$

with

$$\begin{aligned} a &= \frac{b}{2} - \rho \\ b &= 2\mu + 1. \end{aligned} \tag{27}$$

Then w is expressed as ${}_1F_1(a, b, \xi)$ and in order to furnish normalizable ϕ, the confluent hypergeometric function must be a polynomial. This demands that $a = -n$, where n is a nonnegative integer in such a way that

$$\mu = \frac{mc^2}{\lambda} - n, \tag{28}$$

and ${}_1F_1(a, b, \xi)$ is proportional to the associated Laguerre polynomial $L_n^{2\mu}(\xi)$, a polynomial of degree n. This requirement, combined with (21) and (27), implies into quantized energy eigenvalues:

$$E_n = \pm\lambda\sqrt{n\,(2\mu + n)} \tag{29}$$

Finally, we are ready to present the solution for the upper component of the Dirac spinor:

$$\phi_n = N \xi^{\mu} e^{-\xi/2}\, L_n^{2\mu}(\xi)\,, \tag{30}$$

The solution for the lower component, obtained by inserting (30) into (5) and using recurrence relations among the associated Laguerre polynomials [31], is given by

$$\chi_n = \pm\sqrt{\frac{2\mu + n}{n}}\, N \xi^{\mu} e^{-\xi/2} L_{n-1}^{2\mu}(\xi)\,. \tag{31}$$

Since $\mu > 0$, one can conclude from (28) that there is a finite number of bound-state solutions. Further, the components of the Dirac spinor has a

nodal structure in such a way that the number of nodes of the lower component is given by $n-1$. These facts imply that the quantum number n, for $E \neq 0$, is given by $n = 1, 2, 3, \ldots < mc^2/\lambda$. Note that $\lambda < mc^2$ in order that the potential holds at least one excited-state solution.

Therefore, the solution of the problem, including the zero-eigenmode corresponding to $n = 0$, can be written more succinctly as

$$\begin{aligned} E_n &= \pm\lambda\sqrt{n\left(2\frac{mc^2}{\lambda} - n\right)}, \quad n = 0, 1, 2, \ldots < mc^2/\lambda \\ \psi_n &= N_n \xi^{\mu} e^{-\xi/2} \begin{pmatrix} L_n^{2\mu}(\xi) \\ \\ R_n \, L_{n-1}^{2\mu}(\xi) \end{pmatrix}, \end{aligned} \tag{32}$$

where

$$R_n = \begin{cases} 0 & , \text{ for } \; n = 0 \\ \\ \pm\sqrt{\frac{2\mu+n}{n}} & , \text{ for } \; n \neq 0. \end{cases} \tag{33}$$

The top line of Eq. (32), implies that the Dirac energy eigenvalues are restricted to the range $|E| < mc^2$. The energies belonging to $|E| > mc^2$ correspond to the continuum. This conclusion could also be obtained from the definition of μ in (21). In order to evaluate the normalization constant N_n we are confronted by the integral

$$I_n = \int_0^{\infty} d\xi \, \xi^{2\mu-1} e^{-\xi} \left(L_n^{2\mu}\right)^2. \tag{34}$$

One might think that the singularity of the integrand in (34) could be a source of embarrassment. However, the integral is convergent for $\mu > 0$, as it is the case in hands, although it is not so easy calculate it. Fortunately, it has been already done in Refs. [25]-[26]. There results that

$$I_n = \frac{\Gamma(2\mu + n + 1)}{2\mu \, \Gamma(n+1)}. \tag{35}$$

It takes a little algebra to conclude that the normalization constant can be compactly written as

$$N_n = \sqrt{\frac{\lambda/(\hbar c)}{I_n + R_n^2 I_{n-1}}}. \tag{36}$$

In order to get further confidence on the normalizability of the Dirac eigenspinor as well as to get a better understanding of its behaviour , a few plots for $|\phi|^2$ and $|\chi|^2$ are presented. Fig. 1 illustrates the behaviour of the position probability density, $|\psi|^2 = |\phi|^2$, for the zero-mode solution. Figs. 2 and 3 illustrate the behaviour of the upper and lower components of the Dirac spinor, $|\phi|^2$ and $|\chi|^2$, and the position probability density, $|\psi|^2 = |\phi|^2 + |\chi|^2$, for the positive-energy solutions of the first- and second-excited states, respectively. The results for negative energies are the same as far as the charge conjugation does $\phi \rightarrow \phi^*$ and $\chi \rightarrow -\chi^*$. The parameters were chosen for furnishing just three bounded solutions. Note the position probability density has a lonely hump for the omnipresent zero-eigenmode solution. The existence of excited states, though, depends on the relation between the parameter λ and m as given by the top line of Eq. (32).

We have found the solutions of the Dirac equation in 1+1 dimensions for massive fermions coupled to a time-independent potential. Although the coupling is of a Lorentz scalar nature we note that due to its coordinate dependence the potential can not be classified as a true Lorentz scalar. Nevertheless the analysis carried out here can be illuminating to understand the conditions under which a scalar potential can hold bound states, the existence of zero-energy eigenstates as well as its connection to the nonrelativistic and ultrarelativistic limits of the Dirac equation. We also note that the potential we have been dealing with always has the zero-mode eigenstate as a localized solution for $A < 0$, regardless the values of the parameters A and λ, and it can be considered as the ground-state solution. On the other hand, the existence of a finite sequence of excited states depends on the relation between the parameter λ and the mass of the fermion.

Furthermore, the conditions for the existence of a zero-eigenmode for a general scalar potential, equations (12)-(15), including those ones for the massless case, are more general than that one found by Jackiw and Rebbi [32], because we did not restrict ourselves to a solitonic topological scalar field. These last configurations have been shown to be responsible for the induction of fractional fermion number on the vacuum in the second quantization scenario in 1+1 dimensions and the zero-mode of the corresponding Dirac operator plays a fundamental role on this phenomenon [33]. The fermion number fractionization in quantum field theory, as well as the role played by the zero-mode and the continuous spectrum is under investigation by considering the interaction of fermions with nontopological scalar and pseudoscalar backgrounds and will be reported elsewhere.

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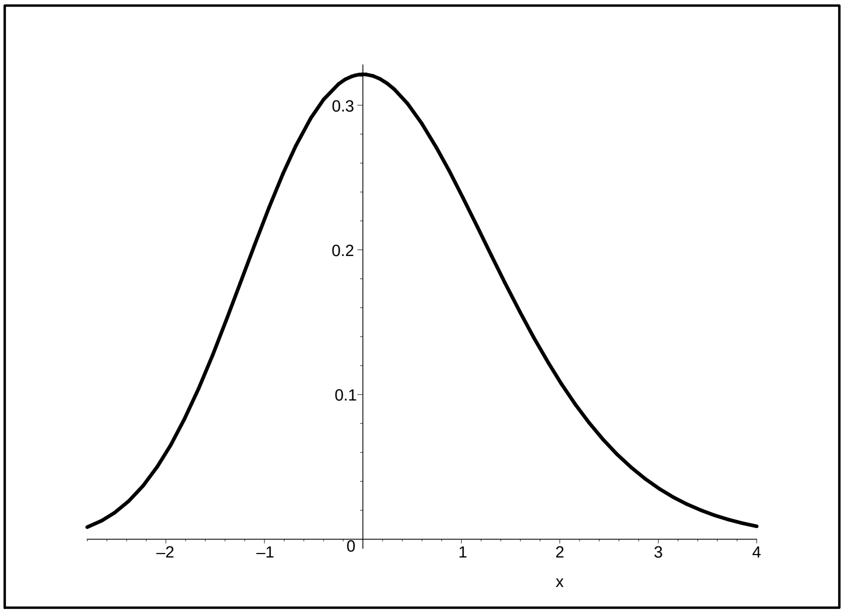

Figure 1: $|\psi|^2 = |\phi|^2$ as a function of x corresponding to the ground state $(E = 0)$ of the potential $V(x) = A \exp\left(-\frac{\lambda}{\hbar c}x\right)$ $(m = \hbar = c = 1,\ A = -1$ and $\lambda = 1/3)$.

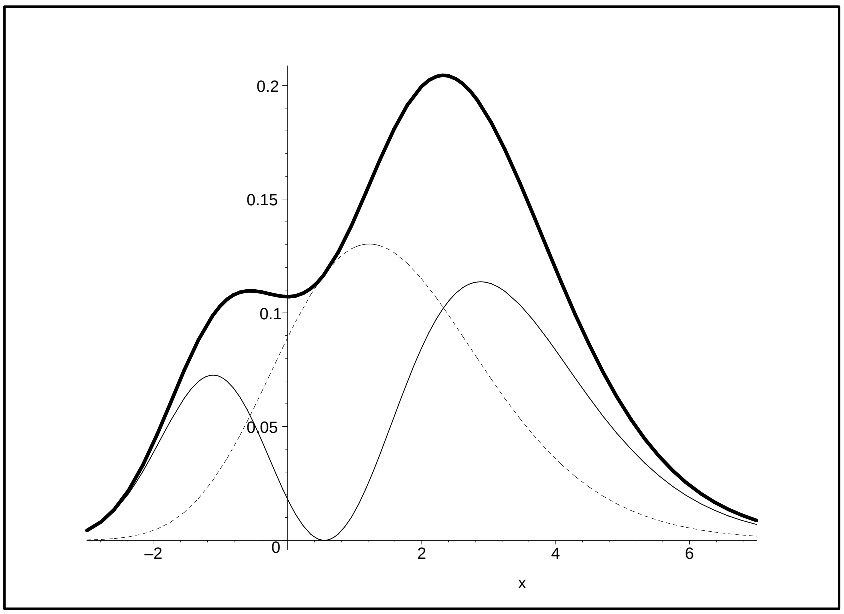

Figure 2: $|\phi|^2$ (full thin line), $|\chi|^2$ (dashed line) and $|\psi|^2 = |\phi|^2 + |\chi|^2$ (full thick line) as a function of x for the first-excited state of the potential $V(x) = A\exp\left(-\frac{\lambda}{\hbar c}x\right)(m = \hbar = c = 1,\ A = -1$ and $\lambda = 1/3)$.

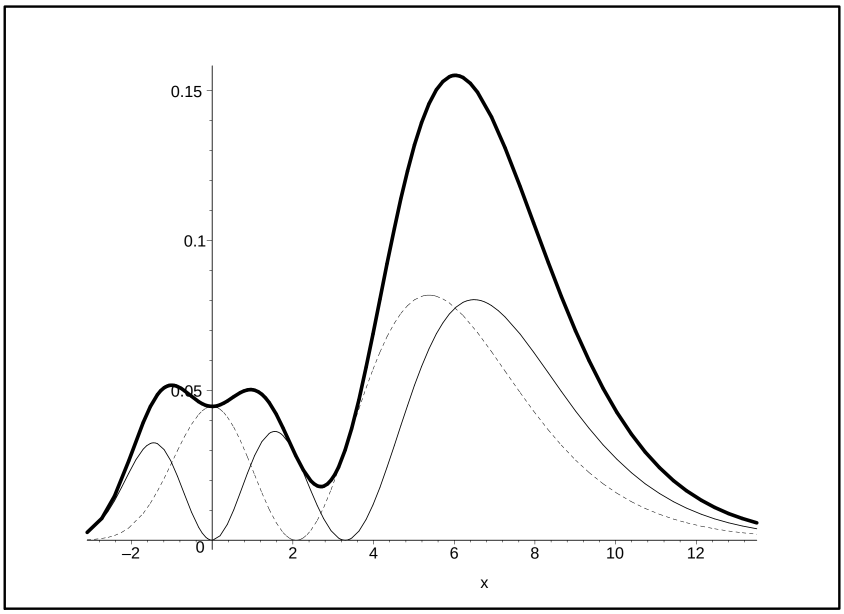

Figure 3: The same as in Fig. 2 for the second-excited state.